ISRAEL JOURNAL OF MATHEMATICS, Vol. 28, Nos. 1-2, 1977

EVERY PLANAR GRAPH HAS AN ACYCLIC 7-COLORING

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ABSTRACT

A proper coloring of the vertices of a graph is said to be acyclic provided that no cycle is two colored. We prove that every planar graph has an acyclic seven coloring.

I. Introduction

A k-coloring of the vertices of a graph is an assignment of one of the colors $1, 2, \cdots, k$ to each vertex so that no two adjacent vertices receive the same color. A k-coloring is said to be *acyclic* if the subgraph induced by the vertices colored with any two colors has no cycle. The *acyclic chromatic number* of a graph G, denoted by $a(G)$, is the minimum value of k for which G has an acyclic k-coloring. The purpose of this paper is to prove the following:

THEOREM. *If G is planar, then $a(G) \leqq 7$.*

In [3], Branko Grünbaum conjectured that if G is planar, $a(G) \leqq 5$. He proved that if G is planar, $a(G) \leqq 9$. John Mitchem [6] proved that if G is planar, $a(G) \leqq 8$. In a forthcoming paper [5], A. V. Kostochka proves, using methods different than ours, that if G is planar, $a(G) \leqq 6$. An analog to the above theorem for toroidal graphs is presented in [1].

The proof is by induction on the number of vertices of G. The theorem is trivially true if G has seven or fewer vertices.

A graph H is said to be reducible if, for any graph G having H as a subgraph, there exists a graph G', having fewer vertices than G, and having the property that any acyclic 7-coloring of G' can be extended to an acyclic 7-coloring of G. In section II we present a list of reducible graphs.

Now, suppose G is a planar graph on the fewest vertices such that $a(G) = 8$.

[+] Research supported in part by the Research Corporation through a Cottrell College Science Research Grant.

Received October 7, 1975 and in revised form October 25, 1976

We assume G is a triangulation; if not, edges may be added to make it so without reducing the acyclic chromatic number. In section III we show that every triangulation of the plane (thus, in particular, G) must contain a reducible graph. This completes the proof of the theorem; for G, being a minimum counter-example, can contain no reducible graph.

II. The reducible graphs

LEMMA 1. *Let G be a planar graph on the fewest vertices such that $a(G) = 8$. G can have no 3-cycle whose interior and exterior are both non-empty.*

PROOF. Suppose that a, b, c is such a 3-cycle. Since the subgraph consisting of the cycle together with its interior has fewer vertices than G, it can be acyclically 7-colored. Assume that a, b and c are colored 1, 2 and 3, respectively.

Likewise, the subgraph consisting of the cycle together with its exterior can be acyclically 7-colored. We may permute the colors in this subgraph so that a, b and c are again colored 1, 2 and 3, respectively. Now this coloring of the exterior vertices may be adjoined to the coloring of the interior vertices to give a 7-coloring of G. A two-color cycle in G can only arise from a two-color cycle in either the interior or the exterior subgraph.

COROLLARY. *A vertex of degree three is reducible.*

LEMMA 2. *A vertex of degree four adjacent to a vertex of degree less than seven is reducible.*

PROOF. Suppose x is a vertex of degree four with neighbors labeled, clockwise, y, a, b and c, with y having degree less than seven. Form G' by deleting x and adding an edge connecting a and c. Acyclically 7-color G', say with a, b and c colored 1, 2 and 3, respectively.

Transfer this coloring to G, with x left uncolored. If y is not colored 2, then x can be made any color not used by a, b, c or y. If y is colored 2, then x can be made whichever of the colors 4, 5, 6, 7 that is not used by the remaining three, or fewer neighbors of y (i.e. other than a, c and x).

All of the remaining reducible graphs must avoid a certain type of 4-cycle. A 4-cycle is said to be *fat* if both the interior and the exterior of the cycle contain more than one point. (We consider the interior to be the region with fewer vertices.)

LEMMA 3. *Suppose H consists of a vertex x of degree five with neighbors labeled clockwise y, a, b, c and d. If y is of degree five or six and if a, x and d are not on any fat 4-cycle, then H is reducible.*

PROOF. Form G' from G by contracting a, x and d to a new vertex x', connected to each point that is connected to a, x or d in G. Acyclically color G', assuming x', b and c are colored 1, 2 and 3, respectively. Transfer this coloring to G, making both a and d color 1, and leaving x uncolored. Note that a and d cannot be adjacent, or else a, x, d is a 3-cycle and we are through by Lemma 1.

If y is not colored 2 or 3, then x can be made any of the three colors not used by neighbors of x. If y is colored 2 or 3, then x can be made whichever of colors 4, 5, 6, 7 that is not used by a neighbor of y.

In either case, the only possible two-color cycles in G must use vertices a, x and d. But suppose $a, x, d, p_1, \cdots, p_n, a$ is a two-color cycle in G. Then $x', p_1, \cdots, p_n, x'$ would be a two-color cycle in G', which cannot be. (Note that p_1 and p_n must be distinct points as a, x and d are on no fat 4-cycle.)

LEMMA 4. *Suppose H consists of a vertex x of degree seven with neighbors labeled clockwise y, a, b, c, z, d and e. If the degree of y and the degree of z are each four or five and if neither y, a and e nor c, z and d are on any fat 4-cycle, then H is reducible.*

PROOF. Form G' from G by contracting y, a and e to a new vertex y' and contracting c, z and d to a new vertex z'. Acyclically color G'. Since x, y', z' and b form a K_4 in G', they must receive four distinct colors, say 1, 2, 3 and 4, respectively. Transfer this coloring to G, with a and e colored 2, c and d colored 3, y and z left uncolored.

Case i. *Assume both y and z are of degree five*

Unless each of y and z has a neighbor (other than x) colored 1, we can assign to each of them any of colors 5, 6, 7 not used by any of its neighbors. For then the only possible two-color cycle containing y would be of the form $e, y, a, \cdots, e$. But this can arise only from a two-color cycle in G' or from a fat 4-cycle. The same argument applies to z. Thus we may assume that each of y and z has a neighbor colored 1.

Color y one of colors 5, 6, 7 that is not used for any of its neighbors. Now color z one of colors 5, 6, 7 that is used neither for any of its neighbors nor for y. Again, this can introduce no two-color cycle.

Case ii. *Assume that y is of degree four*

The reduction in Case i will work if y has no neighbor (other than x) colored 1. If y has a neighbor s which is colored 1; a, x, e, s form a two color cycle. We now recolor x with whichever of colors 5, 6, or 7 is not used for the one or two neighbors of z (other than c, x, and d). Next color y any color not used on any of its neighbors and color z any color not used on any of its neighbors.

LEMMA 5. *Suppose H consists of a vertex x of degree seven with neighbors labeled clockwise y, a, b, c, d, z, and e. If the degree of y and the degree of z are each four or five and if no three of a, d, x, and e are on any fat 4-cycle, then H is reducible.*

PROOF. Form G' from G by contracting a, y, e, z, and d to a new vertex y'. Acyclically 7-color G'. Since x, y', b, and c form a K_4 in G' they must receive four distinct colors, say 1, 2, 3, and 4, respectively. Transfer this coloring to G with a, e, and d colored 2, and y and z left uncolored.

The remainder of the proof of Lemma 5 is identical with Cases (i) and (ii) in the proof of Lemma 4.

REMARK. The preceding lemmas imply that a vertex of degree seven that is on no fat 4-cycle and that is adjacent to two vertices of degree five or less is reducible.

III. Every triangulation contains a reducible graph

This will be proved by the following two lemmas.

LEMMA 6. *Suppose G is a triangulation of the plane having at least four vertices. Then at least one of the following holds:*

(i) *G contains a vertex of degree three,*

(ii) *G contains a vertex of degree less than six adjacent to a vertex of degree less than seven,*

(iii) *G contains a vertex of degree seven adjacent to two vertices of degree less than six.*

LEMMA 7. *Let L be the set of vertices interior to a fat 4-cycle C. Suppose H, the subgraph induced by $L \cup C$, is a triangulation of the interior of C. Then at least one of the following holds:*

(i) *L contains a vertex of degree three,*

(ii) *L contains a vertex of degree less than six adjacent to a vertex of L of degree less than seven,*

(iii) *L contains a vertex of degree seven adjacent to two vertices of L each of degree less than six.*

Lemmas 6 and 7 together guarantee that a triangulation contains a reducible graph. For each of the cases of Lemma 6, unless G contains a fat 4-cycle, it contains a reducible graph. But if it contains a fat 4-cycle C, then Lemma 7 guarantees that among the points interior to C there is either a reducible graph or else a fat 4-cycle C'. We may then apply Lemma 7 to the points interior to C'.

Since we began with a finite graph, this process must eventually terminate with a fat 4-cycle that contains no interior fat 4-cycle and that therefore contains a reducible graph.

Lemmas 6 and 7 can be proved by straightforward counting arguments, but we prefer to use the method of vertex discharging introduced by Heesch [4]. An excellent account of the method is contained in Appel and Haken's work on geographically good configurations [2].

PROOF OF LEMMA 6. Suppose G is a triangulation of the plane with V vertices and E edges. Let $q(x)$ be a function which assigns to each vertex x in G the "charge" $6-$degree of x. Clearly

$$\sum_{x \text{ in } G} q(x) = \sum_{x \text{ in } G} (6 - \text{degree of } x) = 6V - 2E = 12.$$

Assume that G satisfies none of (i), (ii), or (iii). We construct a new function $q'(x)$ which will be obtained from $q(x)$ by "discharging" those vertices in G of degree four or five. If x is a vertex in G of degree four, x will be discharged by sending a charge of $\frac{1}{2}$ to each of its neighbors. If x is a vertex in G of degree five, x will be discharged by sending a charge of $\frac{1}{5}$ to each of its neighbors. Since no vertex of degree four or five can be adjacent to a vertex of degree six or less (we assumed (ii) does not hold), $q'(x) = 0$ if x has degree four, five, or six. If x has degree seven at most one of its neighbors has discharged into x so $q'(x) < 0$. If x has degree K ($\geqq 8$) then x has no more than $K/2$ neighbors discharging into x, hence $q'(x) \leqq 6 - K + K/4 \leqq 0$. Since charge has been conserved in this discharging procedure

$$12 = \sum_{x \text{ in } G} q(x) = \sum_{x \text{ in } G} q'(x) \leqq 0.$$

The contradiction implies that G must satisfy at least one of (i), (ii), or (iii).

PROOF OF LEMMA 7. Suppose H is the smallest graph satisfying the hypotheses of the lemma such that none of (i), (ii), or (iii) hold. We claim H can have no vertices of degree three. If a vertex on C had degree three then there is a 4-cycle C' interior to C. If C' is fat then H is not the smallest counter-example to the lemma. If C' is not fat then C' has in its interior no more than one vertex. If C' has no vertices in its interior, C was not fat. If C' has one vertex in its interior then (ii) must be satisfied.

As in the proof of Lemma 6, we assign each vertex in H a charge given by the function $q(x) = 6 -$ degree of x. Since H is one edge lacking from being a triangulation

$$\sum_{x \text{ in } H} q(x) = \sum_{x \text{ in } H} (6 - \text{degree of } x) = 6V - 2E = 14.$$

We next discharge H using the same rules as in the proof of Lemma 6 except the vertices of C are not discharged. As above, $q'(x) \leqq 0$ if x is in L. By conservation of charge

$$\sum_{x \text{ in } C} q'(x) \geqq 14$$

which implies that for some z in C, $q'(z) \geqq 7/2$. Suppose such a vertex z has degree K in H. No more than $K/2$ of z's neighbors can discharge into z without having (ii) be satisfied. Thus

$$\tfrac{7}{2} \leqq q'(z) \leqq 6 - K + K/4$$

which implies $K \leqq 10/3$. Again the contradiction implies that L must satisfy at least one of (i), (ii), or (iii).

REMARK. Only minor modifications in the proof of Lemma 6 are needed to strengthen condition (iii) to:

(iii)′ G contains a vertex of degree seven adjacent to at least three vertices of degree less than six.

Also, Lemma 7 can be proved with (iii)′ replacing (iii) and assuming that C is a K-cycle where $K \leqq 6$.

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